THE RICCI FLOW ON RADIALLY SYMMETRIC $\mathbb{R}^3$

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Introduction

The Ricci flow $\partial g/\partial t = -2\mathrm{Ric}(g)$ is a way of evolving a Riemannian metric on a smooth manifold. In this paper we will be interested in its behaviour for some complete metrics on $\mathbb{R}^3$. We begin by summarizing what is known about the Ricci flow for complete metrics on non-compact manifolds.

The fundamental work in this area is that of W.-X. Shi [5,6]. The short-time existence theorem, curvature estimates, and many technical results that are used in this paper are due to him. In [6], Shi also proved that if the initial metric is complete, with sectional curvature that is positive, bounded, pinched, and falls off like $1/r^{2+\epsilon}$, the solution $g(t)$ exists for all $t \geq 0$ and converges to a flat metric. Despite Shi's many contributions to the subject, our knowledge of the geometry of complete manifolds of positive curvature has not enabled us to furnish examples of metrics that fulfill these conditions.

Some special results are available in dimension two. In [7], L.-F. Wu shows that for complete metrics on $\mathbb{R}^2$ that are conformal to the standard flat metric, have positive curvature, and satisfy certain boundedness conditions, the Ricci flow exists for all $t \geq 0$ and converges either to the flat metric, or to a non-flat, rotationally symmetric metric which is a soliton metric for the flow. (A "soliton" is a similarity solution to the flow, where the metric is evolving only by diffeomorphisms. A soliton metric is an initial condition that gives rise to such a solution; the appropriate condition for the initial metric is that, for some vector field V, the Lie derivative $\mathcal{L}_V g$ coincide with the Ricci tensor of g.) Which limit prevails depends on whether or not the metric has finite "circumference at infinity", that is, whether or not the ball of radius r has bounded circumference as $r \to \infty$. The soliton metric on $\mathbb{R}^2$, which is $(dx^2 + dy^2)/(1 + x^2 + y^2)$, has finite circumference at infinity.

By assuming g is rotationally symmetric, one can show that there also exists a complete soliton metric on $\mathbb{R}^n$, for $n \geq 3$, which is unique up to homothety. While this metric cannot be written down explicitly, one can compute that it has strictly positive sectional curvature that falls off like $1/r$. (The calculations involved were made for $n = 3$ by R. Bryant, and were subsequently generalized in [4].) We will give sufficient conditions for a rotationally symmetric metric on $\mathbb{R}^3$ to give rise to a solution of the Ricci flow defined for all $t \geq 0$, and for that solution to converge to this soliton metric.

Before stating our main results, we need to set up some notation. By a rotationally symmetric metric on $\mathbb{R}^3$ we mean one for which the rotation group $SO(3)$ acts by isometries. Away from the origin, such a metric can be written as a warped product

$$ds^2 = dr^2 + f(r)^2 d\sigma^2, \tag{1}$$

where r is the metric distance to the origin and $d\sigma^2$ is the standard (constant curvature $+1$) metric on S^2. The metric is smooth at the origin if $f(r)$ extends to be a smooth odd function of r with $f'(0) = 1$. (We will always assume this is the case.) We will let $\nu_1(r)$ be the sectional curvature for the planes that are tangent to the spheres, and let $\nu_2(r)$ be the sectional curvature for the planes that contain a radial direction. In terms of $f(r)$,

$$\nu_1 = \frac{1-(f')^2}{f^2}, \qquad \nu_2 = \frac{-f''}{f}. \tag{2}$$

(The prime will always denote the derivative with respect to r.)

In dimension three, the Ricci flow preserves both positive sectional curvature and positive Ricci curvature. In higher dimensions, it preserves positivity of the curvature operator, that is, of the Riemann curvature tensor considered as a quadratic form on the exterior power $\Lambda^2 TM$ of the tangent bundle. (In dimension three, this is the same as positive sectional curvature; see [1] for details.) With our assumption of rotational symmetry, the eigenvalues of the curvature operator are ν_1 with multiplicity one and ν_2 with multiplicity two, and the scalar curvature R is equal to $2\nu_1 + 4\nu_2$.

Finally, the analysis of the ordinary differential equations that arise in solving for a rotationally symmetric soliton is made easier by the presence of a first integral that is derived from the second Bianchi identity (see [4] for details). This quantity, given in terms of our notation by

$$Q = 2\nu_1 + 4\nu_2 + \left(\frac{f\nu_1 + f\nu_2}{f'}\right)^2, \tag{3}$$

is constant for the rotationally symmetric soliton.

Theorem 1. *Let g be a complete rotationally symmetric metric on $\mathbb{R}^3$ with*

(i) $\nu_1, \nu_2 > 0$ *and* $\nu_1, \nu_2 \le k_0$

(ii) $\nu_2 \le Z\nu_1$

(iii) $\liminf_{r\to\infty} \frac{d}{dr} f^2 > 0$

for positive constants k_0, Z. Then there exists a smooth complete solution to the Ricci flow, with initial metric g, defined for $0 \le t < \infty$.

We will explain where the conditions in Theorem 1 come from. Our assumption of bounded, positive curvature enables us to use Shi's short-time existence theorem, and to control the injectivity radius when it comes time to obtain convergence. We can interpret the other conditions in terms of the behaviour of $f(r)$.

From the formulas for ν_1 and ν_2, and by completeness, the positivity of the curvature implies that $f'(r)$ is monotone decreasing, with

$$0 < f'(r) < 1 \tag{4}$$

for $r > 0$. By simplifying the formula for ν_2/ν_1, one can check that, when condition (i) holds, condition (ii) is equivalent to $\sup |ff''| < \infty$. Condition (iii) implies by integration that f^2 is bounded below by a linear function of r—in other words, that the manifold opens up at least as fast as a paraboloid.

Condition (iii) can also be interpreted in terms of Q. By (3), when condition (i) and condition (ii) hold, $\sup Q < \infty$ if and only if $\sup \dfrac{f\nu_1}{f'} < \infty$. Then (4) and the identity

$$\frac{f\nu_1}{f'} + \frac{f'}{f} = \frac{1}{ff'} \tag{5}$$

show that this is equivalent to $\sup\limits_{f\geq 1} (ff')^{-1} < \infty$. Because Q satisfies a particularly nice heat equation, we will exploit this equivalence in the proof of Theorem 1.

Theorem 2. *Assume the initial metric satisfies the conditions in Theorem 1, and has $Q \geq Q_0$ for a positive constant Q_0. Then $Q(x,t) \geq Q_0$ for all $t \geq 0$.*

Since, under the conditions of Theorem 1, $\inf Q > 0$ is equivalent to $\sup \frac{d}{dr} f^2 < \infty$ (see §2 for details), Theorem 2 says that if the manifold opens up like a paraboloid, the metric does not become flat as time goes on. (For example, by (3), $R \geq Q_0$ at the origin.)

We will obtain a limit as $t \to \infty$, in subsequence, using R. Hamilton's compactness theorem [3], and show the limit is a soliton using Hamilton's theorem on eternal solutions to the Ricci flow [2].

Theorem 3. *Assume the initial metric satisfies the conditions in Theorem 1, and let $g(t)$ be the solution to the Ricci flow. Then there is a sequence of times $t_k \to \infty$, open sets U_k and V_k in $\mathbb{R}^3$, containing the origin, with the U_k exhausting $\mathbb{R}^3$ (i.e. every compact set is contained in U_k for k large enough), diffeomorphisms $\phi_k : U_k \to V_k$ that fix the origin, and a complete metric $\bar{g}$ on $\mathbb{R}^3$, such that (i) on any compact set, the metrics $\phi_k^*(g(t_k))$, together with all their derivatives, converge uniformly to $\bar{g}$; (ii) if $\bar{g}$ is not flat, it is a rotationally symmetric Ricci soliton.*

(Note that the limit $\bar{g}$ will not be flat if the conditions of Theorem 2 are satisfied.)

Example. We conclude this section by constructing an initial metric that satisfies the hypotheses of Theorem 1 and Theorem 2. Let

$$f(r) = r \sin\left(\frac{\pi}{\sqrt{4+|r|}}\right).$$

It is easy to check that (a) f is a C^3 odd function of r, with $\lim_{r\to\infty} f(r) = \infty$; (b) $f'(0) = 1$ and $0 < f'(r) < 1$ elsewhere; (c) $f''(r) < 0$ for $r > 0$, with $\lim_{r\to 0} f''/f = -3\pi^2/256$; (d) $\lim_{r\to\infty} ff' = \pi^2/2$; and (e) $\lim_{r\to\infty} ff'' = 0$.

Next, let

$$\bar{f}(r) = \int_{-\infty}^{\infty} f(\rho)s(r-\rho)d\rho,$$

where $s(r)$ is a C^∞ even function which is positive on the interval $(-\epsilon, \epsilon)$ and zero elsewhere. Then $\bar{f}(r)$ is a C^∞ odd function.

By (c), we can assume that $f'''(r) < 0$ for $r \in (-\delta, \delta)$. Because

$$\bar{f}''(r) = \int_{-\epsilon}^{\epsilon} f''(r+\rho)s(\rho)d\rho = \int_0^{\epsilon} (f''(r+\rho) + f''(r-\rho))s(\rho)d\rho,$$

we make $\epsilon < \delta/2$, and get $\bar{f}''(r) < 0$ for $r > 0$. We also scale $s(r)$ so that $\bar{f}'(0) = 1$.

After checking that $\lim_{r\to 0} \bar{f}''(r)/\bar{f}(r) < 0$, we know that $dr^2 + \bar{f}(r)^2 d\sigma^2$ is a complete metric with strictly positive sectional curvature; let $\bar{\nu}_1$ and $\bar{\nu}_2$

be the sectional curvatures for this metric, defined as in (2). We already know $\bar{\nu}_1$ is bounded. Moreover, since $-f''/f \le C$ for some $C > 0$, then

$$\bar{f}''(r) \ge \int_{-\epsilon}^{\epsilon} -Cf(r+\rho)s(\rho)d\rho = -C\bar{f}(r)$$

when $r \ge \epsilon$, so $\bar{\nu}_2 \le C$.

Let $C_1 = \int_0^\epsilon s(\rho)d\rho$. Then for $r \ge \epsilon$,

$$\begin{aligned}
\bar{f}(r) &\ge \int_0^{\epsilon} f(r)s(\rho)d\rho = C_1 f(r), \\
\bar{f}(r) &\le \int_{-\epsilon}^{\epsilon} (|f(r+\rho) - f(r)| + f(r))s(\rho)d\rho \le 2C_1(\epsilon + f(r)).
\end{aligned}$$

Using these estimates, and the fact that f is monotone, one can show without difficulty that $\liminf_{r\to\infty} ff' > 0$ implies $\liminf_{r\to\infty} \bar{f}\bar{f}' > 0$, that $\sup|ff''| < \infty$ implies $\sup|ff''| < \infty$, and that $\sup|ff'| < \infty$ implies $\sup|\bar{f}\bar{f}'| < \infty$. (These are condition (iii) in Theorem 1, an equivalent for condition (ii), and an equivalent for the condition on Q in Theorem 2, respectively.) Details are left to the reader.

1. Long-Time Existence

Our proof of long-time existence depends on showing there exists a T such that, for $t \in [0,T]$, $\sup Q(\cdot,t) \le \sup Q(\cdot,0)$. (The dot indicates a supremum over all $\mathbb{R}^3$.) This is done by showing that $\sup Q(\cdot,t)$ remains finite up to time T, and then applying to maximum principle; the finiteness of $\sup Q$ will follow from an estimate on $|ff'''|$. Once we have a time-invariant upper bound for Q, it gives by (3) a time-invariant upper bound for the curvature, which then allows us to iterate the short-time existence theorem arbitrarily many times.

Theorem 1.1. (Short-time existence, [5]) *Suppose (M^n, g) is a noncompact complete Riemannian manifold with sectional curvature bounded in absolute value by $k_0 > 0$.*

(i) There exists a smooth complete solution $g(t)$ to the Ricci flow, with $g(0) = g$, defined for $0 \le t \le T(k_0)$.

(ii) There are positive constants C_0, C_1, C_2, ... , depending on k_0 and n, such that the ℓth covariant derivative of the curvature satisfies $|\nabla^\ell \mathrm{Rm}|^2 \le C_\ell / t^\ell$ when $0 < t \le T(k_0)$.

Next, we will review the evolution equations in the rotationally symmetric case. Here, the Ricci flow becomes a parabolic equation for f as a function of time and one 'space' variable. To compute this equation one has to fix a time-invariant radial coordinate ρ (since the distance to the origin is changing under the flow) and let the metric be $h^2 d\rho^2 + f^2 d\sigma^2$. Then the Ricci flow implies

$$\partial_t f = -(\nu_1 + \nu_2) f, \qquad \partial_t h = -2\nu_2 h.$$

The first equation shows that under positive curvature the Ricci flow shrinks the $SO(3)$ orbits. The second equation shows us how we have to commute the operators $\partial_t = \partial/\partial t$ and $\partial_r = \partial/\partial r$. For, if F is a function of ρ and t,

$$\partial_t \partial_r F = \partial_t \left(\frac{1}{h} \frac{\partial F}{\partial \rho} \right) = \partial_r \partial_t F + 2\nu_2 \partial_r F.$$

This allows us to compute the evolution equations of quantities involving the r-derivatives f', f'' and f'''. To compute the heat operator $\partial_t - \triangle$ on such quantities, we also need the formula $\triangle = \partial_r^2 + 2f'/f \partial_r$ for the Laplacian in radial coordinates.

With these formulas, one can compute that

$$\begin{aligned} (\partial_t - \triangle)\nu_1 &= 2\nu_1^2 + 2\nu_2^2 + 4(f')^2(\nu_2 - \nu_1)/f^2 \\ (\partial_t - \triangle)\nu_2 &= 2\nu_2^2 + 2\nu_1\nu_2 + 2(f')^2(\nu_1 - \nu_2)/f^2 \end{aligned}$$

and

$$\begin{aligned} (\partial_t - \triangle) Q = &- \frac{(ff')^2}{(1 + (f')^2 + f\nu_2)^2} \left(\frac{\partial Q}{\partial r} \right)^2 \\ &- 2 \left(\frac{f\nu_2}{f'} + \frac{(f')^2}{1 + (f')^2 + f\nu_2} \left(\frac{f\nu_1 + f\nu_2}{f'} \right) \right) \frac{\partial Q}{\partial r}. \end{aligned} \tag{6}$$

Now we can show that, if Z in Theorem 1 is large enough, condition (ii) persists.

Lemma 1.2. *Assume $Z > 1$ and $2Z^2 - 2Z - 1 > 0$. Under the assumptions of Theorem 1, $\nu_2 \leq Z\nu_1$ holds for all $t \in [0, T(k_0)]$.*

Proof. Compute that

$$\begin{aligned}(\partial_t - \triangle)(\nu_2 - Z\nu_1) &= -2(Z-1)(\nu_2 - Z\nu_1)^2 \\ &\quad - 2\frac{(\nu_2 - Z\nu_1)}{f^2}((f')^2(1+2Z) + f^2\nu_1(Z^2 - 2Z - 1)) \\ &\quad - 2(Z-1)\nu_1(Z^2\nu_1 + (1+2Z)(f'/f)^2).\end{aligned}$$

Since $\nu_2 - Z\nu_1$ is bounded, and the right hand side is nonpositive if $\nu_2 \geq Z\nu_1$, we can apply the maximum principle ([6], Lemma 4.5). $\square$

In the course of estimating ff''', we will need a compact set outside of which $f \geq 1$.

Lemma 1.3. *Under the assumptions of Theorem 1, for any $\beta > 0$ there is a compact set K such that $f^2 > \beta$ outside K for all $t \in [0, T(k_0)]$.*

Proof. Using the curvature bounds from Theorem 1.1,

$$\partial_t f^2 = -2(\nu_1 + \nu_2)f^2 \geq -4C_0 f^2.$$

So, it is enough to pick K so that $f^2 \geq e^{4C_0T(k_0)}\beta$ outside K at time zero. $\square$

We will also need to use Shi's smoothing of the distance function, and the bounds on its derivatives.

Lemma 1.4. ([6], Lemmas 4.1–4.3) *Assume $g(t)$, $t \in [0, T]$, is a solution of the Ricci flow on a noncompact manifold M, with $g(t)$ complete, such that the sectional curvature is strictly positive and uniformly bounded on $M \times [0, T]$, and $\int_0^T |\nabla \mathrm{Rm}| dt \leq C$ on M. Then for any $x_0 \in M$ there exists* *a function ψ on M such that*

$$c_1(1 + d_0(x_0, x)) \leq \psi(x) \leq c_2(1 + d_0(x_0, x)),$$
$$|\nabla \psi|^2 \leq c_3,$$
$$\triangle \psi \leq c_4,$$

where d_0 is the distance measured using $g(0)$, the bounds on the norm of $\nabla \psi$ and on the Laplacian hold when computed using $g(t)$, and c_1, c_2, c_3, c_4 are positive constants that depend only on C, T, the dimension of M, and the bounds for curvature.

Lemma 1.5. *Let g satisfy the conditions in Theorem 1, and let $g(t)$ be the solution of the Ricci flow with $g(0) = g$, provided by Theorem 1.1, defined for $t \in [0, T(k_0)]$. Then for any $x_0 \in \mathbb{R}^3$, there is a function $\tilde{\psi}$ and positive constants c_3, c_5, and $\rho_1 < \rho_2$ such that $(\partial_t - \triangle)\tilde{\psi} \geq 0$, $|\nabla \tilde{\psi}|^2 \leq c_3$, and $\tilde{\psi}(x,t) < c_5$ for $x \in \mathcal{B}_0(x_0, \rho_1)$ and $\tilde{\psi}(x,t) > 2c_5$ outside $\mathcal{B}_0(x_0, \rho_2)$, for all $t \in [0, T(k_0)]$. (Here, $\mathcal{B}_0$ means the geodesic ball with respect to $g(0)$.) Moreover, the constants depend only on k_0.*

Proof. From Theorem 1.1, $\int_0^{T(k_0)} |\nabla \mathrm{Rm}| dt \leq C$ for C depending only k_0. Then let ψ be as in Lemma 1.4, where c_1, c_2, c_3, c_4 now depend only on k_0. Let $\tilde{\psi} = \psi + c_4 t$, which gives $(\partial_t - \triangle)\tilde{\psi} \geq 0$ and $|\nabla \tilde{\psi}|^2 \leq c_3$. To get $\tilde{\psi} < c_5$ on $\mathcal{B}_0(x_0, \rho_1)$, we need $c_5 > c_2(1 + \rho_1) + c_4 t$. To get $\tilde{\psi} > 2c_5$ outside $\mathcal{B}_0(x_0, \rho_2)$, we need $2c_5 < c_1(1 + \rho_2)$. So, choose ρ_1, ρ_2 so that $c_2(1 + \rho_1) + c_4 T(k_0) < c_1(1 + \rho_2)/2$, and let c_5 be some number in between. □

Lemma 1.6. *Under the assumptions of Theorem 1, there is a constant C such that $|ff'''| < C/\sqrt{t}$ on $M \times [0, T(k_0)]$.*

Proof. Because $\nu_2 \leq Z\nu_1$, we know $(ff'')^2$ is bounded; ff''' shows up in its evolution equation:

(7)

$$(\partial_t - \triangle)(ff'')^2 = 4\nu_2(ff'')^2 - 2(ff''')^2 - 12ff'f''f''' + 4(ff')^2\nu_1\nu_2 - 10(f'f'')^2.$$

We will use this to offset the term in $(ff''')^2$ in the heat equation

$$(\partial_t - \triangle)(ff''')^2 = -2(\partial_r(ff'''))^2 - 8f'f'''\partial_r(ff''') + 12\nu_2(ff''')^2 \\ + f'f'''\left((12 - 28(f')^2)\nu_2 - 8(f'')^2 + 12(f')^2\nu_1\right). \tag{8}$$

First, we get rid of the $ff'f''f'''$ term in (7) by completing the square (i.e. $|2ab| \leq \epsilon a^2 + b^2/\epsilon$ for any $\epsilon > 0$). The remaining terms can be bounded using $f' \leq 1$, $f^2\nu_1 < 1$, $\nu_2 \leq k_0$, and

$$(f'f'')^2 = (ff')^2\nu_2^2 \leq (Zf\nu_1)^2 \leq Z^2(1-(f')^2)\nu_1 \leq Z^2C_0.$$

So, there is a constant B_0 depending on k_0 and Z such that

$$(\partial_t - \triangle)(ff'')^2 \leq -(ff''')^2 + B_0.$$

Next, we work on (8). By Lemma 1.3, there is a compact set K such that $f \geq 1$ outside K at any time $t \in [0, T(k_0)]$. Using $|f'f'''| \leq |ff'''|$ and $|f''| \leq |ff''|$ outside K, and completing squares, we get

$$(\partial_t - \triangle)(ff''')^2 \leq -(\partial_r(ff'''))^2 + B_1(ff''')^2 + B_2,$$

for constants B_1, B_2 depending on k_0.

Let $\phi = (a + (ff'')^2)(ff''')^2$, where $a > 0$ will be chosen later; then

$$(\partial_t - \triangle)\phi \leq (a + (ff'')^2)(-(\partial_r(ff'''))^2 + B_1(ff''')^2 + B_2) \\ - (ff''')^4 + B_0(ff''')^2 - 2\partial_r(f^2(f''')^2)\partial_r(f^2(f'')^2)$$

Completing the square for the last term, we get

$$8|f^2f''f'''(ff''' + f'f'')\partial_r(ff''')| \leq 4\left(\alpha(\partial_r(ff'''))^2 + \frac{1}{\alpha}(ff'')^2(ff''')^4\right) \\ + 4\left(\beta(\partial_r(ff'''))^2 + \frac{1}{\beta}(ff')^2(f'')^4(ff''')^2\right)$$

for any α, $\beta > 0$. Choose them so that $4(\alpha + \beta) = a$. Since $(ff'')^2 \leq Z^2$, the coefficient of $(ff''')^4$ is now at most $-1 + 4Z^2/\alpha$. If we choose $a = 24Z^2$,

$\alpha = 20Z^2$, $\beta = 5Z^2$, this coefficient is $-4/5$, and $a + (ff'')^2 \le 25Z^2$. Now we have

$$(\partial_t - \triangle)\phi \le -B_3\phi^2 + B_4\phi + B_5$$

for $B_3 = \frac{4}{5}(25Z^2)^{-2}$ and B_4, B_5 depending on k_0 and Z. In fact, by letting $\tilde{\phi} = B_3\phi - B_4/2$, we can get $(\partial_t - \triangle)\tilde{\phi} \le -\tilde{\phi}^2 + b^2$ where $b = \sqrt{B_3B_5 + B_4^2/4}$.

Let K_1 be a compact set such that if x_0 is outside K_1, $\mathcal{B}_0(x_0, \rho_2)$ lies outside K, where ρ_2 is as in Lemma 1.5. Given $x_0 \notin K_1$, construct $\tilde{\psi}$ as in 1.5, and let

$$F(x,t) = \frac{6c_3}{(2c_5 - \tilde{\psi}(x,t))^2} + \frac{b(1 + e^{-2bt})}{1 - e^{-2bt}},$$

where c_3, c_5 are as in Lemma 1.5. Then F is defined at least on $\mathcal{B}_0(x_0, \rho_1)$ when $t > 0$. In fact, since the first term of F is bounded by a constant when $x = x_0$, and

$$\lim_{t \to 0^+} \frac{tb(1 + e^{-2bt})}{1 - e^{-2bt}} = 1,$$

then there is a $C > 0$, independent of x_0, such that $F(x_0, t) \le C/t$ when $0 < t \le T$. Furthermore, F satisfies $(\partial_t - \triangle)F \ge -F^2 + b^2$, so F is a supersolution for the equation for which $\tilde{\phi}$ is a subsolution. Since F grows to infinity between $\mathcal{B}_0(x_0, \rho_1)$ and the boundary of $\mathcal{B}_0(x_0, \rho_2)$, $\tilde{\phi} - F$ must have an interior maximum on $\mathcal{B}_0(x_0, \rho_2) \times (0, T)$, or a maximum on the boundary at $t = T$, and at that point

$$0 \le (\partial_t - \triangle)(\tilde{\phi} - F) \le F^2 - \tilde{\phi}^2.$$

Thus, $F \ge \tilde{\phi}$ at (x_0, t). Then $\tilde{\phi}(x_0, t) \le C/t$ and a similar bound applies to $\phi(x_0, t)$. We get the desired bound on $|ff'''|$ from one on ϕ by noting that $\phi \ge a(ff''')^2$, where a was chosen above to be $24Z^2$. $\square$

Lemma 1.7. *Suppose $Q \le \ell_0$ at time zero. Under the assumptions of Theorem 1, there is a time $\tilde{T}(k_0, \ell_0) > 0$ such that $\sup_{0 \le t \le \tilde{T}} Q(\cdot, t) \le \ell_0$.*

Proof. By (3) and (5),

$$Q \le \ell_0 \Rightarrow \frac{f\nu_1}{f'} \le \sqrt{\ell_0} \Rightarrow \sup_{f \ge 1} (ff')^{-1} \le \sqrt{\ell_0} + 1. \tag{9}$$

Let K be a compact set such that $f \geq 1$ outside K for $t \in [0, T(k_0)]$. Since

$$\frac{\partial}{\partial t}\frac{1}{ff'} = \frac{-ff'''}{(ff')^2} + \frac{\nu_2}{ff'},$$

the estimate on $|ff'''|$ obtained above shows that $(ff')^{-1}$ can be compared to the solutions of an ODE of the form $dy/dt = ay + by^2 t^{-1/2}$, where a, b are positive constants. Although positive solutions of this equation blow up in finite time, it is easy to show that if $y(0) \leq A$ then there is a time $\tau(A, a, b)$ such that $y(t) \leq 2A$ when $t \leq \tau$. We will let $\tilde{T} = \min(\tau, T(k_0))$, with $A = \sqrt{l_0} + 1$; then $\sup_{t \leq \tilde{T},\ x \notin K} (ff')^{-1} < \infty$.

Again, by (3) and (5), this implies that $\sup_{t \leq \tilde{T}} Q(\cdot, t) < \infty$. Then applying the maximum principle to (6) gives the result.□

Proof of Theorem 1. By (3) and the last lemma, $\nu_1, \nu_2 \leq \ell_0$ for $t \in [0, \tilde{T}(k_0, \ell_0)]$. Now apply Theorem 1.1, starting at time $\tilde{T}(k_0, \ell_0)$, with k_0 replaced by ℓ_0. The solution extends for an additional time at least $\tilde{T}(\ell_0, \ell_0)$, and the upper bound $Q \leq \ell_0$, and consequently $\nu_1, \nu_2 \leq \ell_0$, persists over the additional time interval. Because we can repeat this as many times as we like, the solution exists for all $t \geq 0$.□

Corollary 1.8. *For the solution obtained in Theorem 1, the estimates*

$$Q \leq \ell_0, \qquad \nu_2 \leq Z\nu_1$$

hold for all $t > 0$. There is a constant $\widetilde{C_0}$ such that $\nu_1, \nu_2 \leq \widetilde{C_0}$ for all $t \geq 0$. Given any $\tau > 0$ there are constants $\widetilde{C_\ell}(\tau)$, $\ell = 1, 2, 3, \ldots$, such that $|\nabla^\ell \mathrm{Rm}|^2 \leq \widetilde{C_\ell}$ for $t \in [\tau, \infty)$.

Proof. That $Q \leq \ell_0$ and $\nu_2 \leq Z\nu_1$ persist is clear from Lemma 1.2 and the proof of Theorem 1. We can let $\widetilde{C_0} = \max(k_0, \ell_0)$.

When $t \geq \tau$, a time τ has elapsed since the curvatures were bounded above by $\widetilde{C_0}$. Now we apply the estimates in Theorem 1.1, thinking of time $t - \tau$ as time zero, and get a bound C_ℓ/τ for the ℓth covariant derivative of curvature. This will be our $\widetilde{C_\ell}$. □

2. Preserving Non-flatness

Lemma 2.1. *For a rotationally symmetric metric satisfying the conditions of Theorem 1,* $\inf Q > 0$ *is equivalent to* $\sup ff' < \infty$.

Proof. First, by (3)

$$\begin{aligned} Q &\le (2+4Z)\nu_1 + (1+Z)^2 (f\nu_1/f')^2 \\ &= (1+Z)^2 \left(\frac{1}{(ff')^2} - \frac{(f')^2}{f^2} \right) - 2Z^2\nu_1, \end{aligned}$$

which shows that

$$(ff')^2 \le (1+Z)^2/Q. \tag{10}$$

On the other hand,

$$Q \ge \left(\frac{f\nu_1}{f'} \right)^2 + 2\nu_1 = \frac{1-(f')^4}{(ff')^2}. \tag{11}$$

By condition (iii) in Theorem 1, f^2 is bounded below by some linear function of r. So, if $\limsup_{r\to\infty} ff' < \infty$, there is a compact set outside of which $(f')^4 \le 1/2$, and the right hand side of (11) is at least $\frac{1}{2}(ff')^{-2}$. $\square$

Proof of Theorem 2. Since

$$\partial_t (ff') = ff''' + \nu_2 ff',$$

we use the bounds on ν_2 and $|ff'''|$ obtained in §1 to compare ff' to the solution of an ordinary differential equation. Since $\sup ff' < \infty$ at time zero, then $\sup ff' < \infty$ for $t \in [0,T]$. Then, by the above lemma, $\inf_{t\in[0,T]} Q(\cdot,t) > 0$.

In preparation for applying the maximum principle to Q, we need a *lower* bound for the right-hand side of (6). We use (10) to bound the size of the coefficient of $(\partial Q/\partial r)^2$, and we use the bounds on the terms of Q to bound the coefficient of $\partial Q/\partial r$, giving

$$(\partial_t - \triangle) Q \ge -\frac{A}{Q} \left(\frac{\partial Q}{\partial r} \right)^2 - B \left| \frac{\partial Q}{\partial r} \right|,$$

for some positive constants A, B. If $\phi = 1/Q$, then

$$(\partial_t - \triangle)\phi \leq A\left(\frac{\partial\phi}{\partial r}\right)^2 + B\left|\frac{\partial\phi}{\partial r}\right|.$$

Now let ψ, c_3, c_4 be as in Lemma 1.4. Let $\widetilde{\phi} = \phi - \epsilon(\psi + \mu t)$ for some positive ϵ and μ. Then $\widetilde{\phi}$ has only interior maxima, and at one of these

$$\frac{\partial\widetilde{\phi}}{\partial t} \leq \epsilon\left(\epsilon A\left(\frac{\partial\psi}{\partial r}\right)^2 + B|\frac{\partial\psi}{\partial r}| + \triangle\,\psi - \mu\right).$$

So, if we choose μ large relative to A, B, c_3 and c_4, this shows that the maximum of $\widetilde{\phi}$ is nonincreasing. Let $\epsilon \to 0$ to see that the same is true for ϕ.

It follows that $\inf_{t\in[0,T]} Q(\cdot,t) \geq \inf Q(\cdot,0)$. Since T is arbitrary, this proves Theorem 2. $\square$

3. Convergence

We will obtain what might be called "modified subsequential convergence", that is, convergence of the metric along a sequences of times $t_k \to \infty$, provided we are allowed to modify the metric by a sequence of diffeomorphisms. We will resort to Hamilton's compactness theorem for the Ricci flow.

Definition. *Suppose $g(t)$ is a solution to the Ricci flow on M, defined for $t \in (\alpha, \omega)$, where $-\infty \leq \alpha < 0$, $0 < \omega \leq \infty$. Suppose $p \in M$, f is a frame at p, and g is complete at each time. Then (M, g, p, f) is an evolving complete marked Riemannian manifold.*

Theorem 3.1. [3] *Suppose that (M_k, g_k, p_k, f_k) is a sequence of evolving complete marked Riemannian manifolds defined for $t \in (\alpha, \omega)$ such that (a) the absolute values of the sectional curvatures of the g_k are at all times uniformly bounded by a constant B independent of k, and (b) the injectivity radius of $g_k(0)$ at p_k is bounded below by a constant $\delta > 0$ independent of k. Then, once one passes to an appropriate subsequence, there is an evolving* *complete marked Riemannian manifold* $(M, \bar{g}, p, f)$*, open sets* U_k *exhausting* M*, open sets* $V_k \subset M_k$*, and diffeomorphisms* $\phi_k : U_k \to V_k$ *taking* p *to* p_k *and* f *to* f_k*, such that the metrics* $\phi_k^* g_k$ *converge uniformly to* g *on every compact subset of* $M \times (\alpha, \omega)$*, together with all their derivatives.*

Proof of Theorem 3(i). Suppose we pick a sequence t_k of times, with $t_k \geq k$, and let $g_k(t) = g(t + t_k)$, $M_k = \mathbb{R}^3$ and p_k be the origin, with a fixed frame. (We will choose the t_k more carefully later.) By Corollary 1.8, the sectional curvatures are uniformly bounded by $\widetilde{C_0}$. Since the sectional curvatures are also strictly positive, Lemma 3.3 in [6] implies that injectivity radius is uniformly bounded below by $\pi \widetilde{C_0}^{-1/2}$.

Since our g_k is defined for $t \in (-k, \infty)$, for any such interval we can get a limiting solution $\bar{g}(t)$ defined on M.

Lemma 3.2. *The injectivity radius of* $\bar{g}(0)$ *at* p *is infinite.*

Proof. Let $v \in T_p M$, $\ell = |v|_{\bar{g}}$, and $q = \exp_{\bar{g}}(v)$, and suppose

$$d_{\bar{g}}(p, q) \leq \ell - 3\epsilon. \tag{12}$$

Let γ be a $\bar{g}$-geodesic that minimizes distance from p to q. Then for k large enough, $L_{g_k}(\gamma) \leq L_{\bar{g}}(\gamma) + \epsilon$, and hence

$$d_{g_k}(p, q) \leq d_{\bar{g}}(p, q) + \epsilon. \tag{13}$$

(We write g_k for the pullback $\phi_k^* g_k$, and we'll assume that k is large enough so that these pullbacks are defined on a ball of radius 2ℓ about p.)

Now let v_k be the positive multiple of v of length ℓ in the g_k metric. Then $\lim_{k \to \infty} \exp_{g_k}(v_k) = q$; in fact, for k large enough,

$$\ell = d_{g_k}(p, \exp_{g_k}(v_k)) \leq d_{g_k}(p, q) + \epsilon.$$

Coupled with (13), this contradicts (12).□

This allows us to identify M with $\mathbb{R}^3$ via the exponential map. We get a limit $\bar{g}(t)$ on $\mathbb{R}^3$ defined for $t \in (-\infty, \infty)$ by passing to a diagonal subsequence. (While one gets Theorem 3(i) just by letting $\bar{g} = \bar{g}(0)$, we will need $\bar{g}(t)$ to be an "eternal solution", i.e. one defined for all $t \in \mathbb{R}$, for the sequel.)

Proof of Theorem 3(ii). We will show that the t_k can be chosen to produce a limit $\bar{g}$ that is a Ricci soliton by using the following theorem of Hamilton [2]:

Theorem 3.3. *If $g(t)$ is a complete eternal solution of the Ricci flow on a simply-connected manifold, with uniformly bounded curvature and strictly positive curvature operator, and there is a point where the scalar curvature R has a local maximum and has $\partial R/\partial t \leq 0$, then $g(t)$ is a Ricci soliton.*

So far, we know that our limit $\bar{g}$ has bounded curvature and non-negative curvature operator. Let

$$S = \lim_{\tau \to \infty} \sup_{\substack{x \in \mathbb{R}^3 \\ t \geq \tau}} R(x,t),$$

where $R(x,t)$ is the scalar curvature of our original solution $g(t)$ at x. If $S = 0$ then, since the scalar curvature is the sum of the sectional curvatures, $\bar{g}(0)$ must be flat—a trivial soliton metric. So, assume $S > 0$ from now on.

The next two lemmas show that R can only be large in some geodesic ball around the origin.

Lemma 3.4. *Let $g(t)$ be the solution of the Ricci flow provided by Theorem 1. Then there is a constant $C > 0$ such that for all $t \geq 0$,*

$$\frac{(f')^2(1+f^4)}{f^2+(f')^2} \geq C.$$

Proof. Let η stand for the quantity on the left-hand side of this inequality. Clearly η is strictly positive, and since

$$\eta \geq \frac{f^2}{1+f^2}(ff')^2,$$

assumption (iii) in Theorem 1 implies $\liminf_{r\to\infty} \eta > 0$ at time zero. We will apply the maximum principle to see that η has a lower bound independent of t. One computes that

$$(\partial_t - \triangle)\eta = -\frac{(f^2+(f')^2)(f^2-3(f')^2)}{2(f')^2(1+f^4)}\left(\frac{1}{f}\frac{\partial\eta}{\partial r}\right)^2 - 4\frac{(f^2+3(f')^2)ff'}{1+f^4}\frac{\partial\eta}{\partial r} + \frac{4(f')^2(f^2+(f')^2)}{f^6(1+f^4)^2}G(f^2,\eta)$$

where

$$G(x,y) = x^5y+6xy^2+x^2\left[(x^2(1-y)-6xy+(1-y)^2\right]+(1-y^2)(x^2-xy+1).$$

When $0<y<1/2$, the quadratic in square brackets is at least $\frac{1}{2}x^2-6xy+1/4$. Then, to make $G(x,y)$ positive, constrain y so that the discriminants of the two quadratics in x are negative; this is accomplished if $y<1/\sqrt{72}$. Thus, $G(x,y)>0$ when $x\geq 0$ and $0<y\leq 1/10$. By Corollary 1.8 and (9), there is a $\tilde{C}$ such that $\liminf_{r\to\infty}\eta \geq \tilde{C}$ at all times. If η ever has an interior minimum less than both $\tilde{C}$ and $1/10$, $\partial\eta/\partial t \geq 0$ there. Hence η is always bounded below by the least of $\tilde{C}$, $1/10$ and its infimum at time zero. □

Lemma 3.5. *Let $g(t)$ be as in the previous lemma. Given any $\rho>0$, there is a D independent of t such that the scalar curvature $R(x,t)\leq\rho$ outside a ball of radius D about the origin.*

Proof. Since $R = 2\nu_1+4\nu_2 \leq (2+4Z)\nu_1$, it is enough to show that $\nu_1\leq\rho$ when r is large enough. Since $\nu_1 \leq 1/f^2$, it is enough to show that $f^2\geq C$ for r large enough. From the fact that $f(0)=0$, $f''<0$ and $f'>0$, it follows that $f'(r)\leq f(r)/r$. Then

$$\frac{1+f^4}{1+r^2} \geq \frac{(f')^2(1+f^4)}{f^2+(f')^2}.$$

So, by the previous lemma, $1+f^4\geq C(1+r^2)$ for $C>0$ independent of t. □

Let D be the radius provided by the last lemma for some $\rho < S/2$. We choose t_k so that, for k large enough, $R \geq S/2$ somewhere at time t_k; then this must happen within distance D of the origin. So, by taking an accumulation point of the x_k, the scalar curvature of $\bar{g}(0)$ is positive at some point within radius D of the origin. Now, for any solution of the Ricci flow, the scalar curvature satisfies $\partial R/\partial t \geq \triangle R + R^2$. This implies, by the maximum principle, that if, at time t, $R \geq 0$ everywhere $R > 0$ somewhere, then $R > 0$ everywhere at later times. In our case, this immediately shows that $\bar{g}(t)$ has positive scalar curvature for $t > 0$. In fact, $\bar{g}(t)$ must have had positive scalar curvature somewhere at each $t < 0$, because non-negative sectional curvature and zero scalar curvature would imply that $\bar{g}(t)$ is flat, and would remain flat at subsequent times under the Ricci flow. So, we know that $\bar{g}(t)$ has positive scalar curvature everywhere at all times.

Suppose that, at a given time t, $\bar{g}(t)$ has scalar curvature $R_0 > 0$ at the origin. Then for k large enough, $g_k(t)$ has scalar curvature at least $R_0/2$ at the origin. By rotational symmetry, this means the eigenvalues of the curvature operator are at least $R_0/12$ there. Hence $\bar{g}(t)$ has strictly positive curvature operator at the origin, at any given time t. Since $\bar{g}$ already has non-negative curvature operator, we know that $\bar{g}(t)$ has strictly positive curvature operator everywhere at all times (cf. Lemma 8.2 in [1]).

We need to show that the times t_k can be chosen to ensure that, for the limit $\bar{g}(t)$, there is a point where R achieves a local maximum and has $\partial R/\partial t \leq 0$. To do this, take t_k such that (i) $t_k \geq k$, and (ii) for some point x_k within distance D of the origin, R achieves a local maximum at x_k at time t_k, and $R(x_k, t_k) + 1/k^2 \geq \sup_{t>k} R(\cdot, t)$. Then $\partial R/\partial t(x_k, t) \leq 1/k$ when $t \in [t_k, t_k + 1/k]$. Again, since the x_k lie in a geodesic ball of radius D, there is an accumulation point; and, the scalar curvature of $\bar{g}(0)$ satisfies the requirements of Theorem 3.3.

It is easy to see that $\bar{g}$ is rotationally symmetric. Let γ be a rotation of $\mathbb{R}^3$. Because we use the exponential map of $\bar{g}$ at p to identify M with $\mathbb{R}^3$, the metrics $\gamma^* g_k$ converge to $\gamma^* \bar{g}$. Since $\gamma^* g_k = g_k$, then $\gamma^* \bar{g}$ differs from the other limit $\bar{g}$ by an isometry.

This ends the proof of Theorem 3.

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