AFFINE ISOMETRIC EMBEDDING FOR SURFACES

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Abstract A strictly convex hypersurface in $\mathbb{R}^n$ can be endowed with a Riemannian metric in a way that is invariant under the group of (equi)affine motions. We study the corresponding isometric embedding problem for surfaces in $\mathbb{R}^3$. This problem is formulated in terms of a quasilinear elliptic system of PDE for the Pick form. A negative result is obtained by attempting to invert about the standard embedding of the round sphere as an ellipsoid.

1. Introduction

A strictly convex hypersurface M^n in $\mathbb{R}^{n+1}$ can be given a Riemannian metric in a way that is invariant under affine motions, i.e., the action of $SL(n+1)$ and translations. Essentially, if f is the position in $\mathbb{R}^{n+1}$ as a function of local coordinates on M, let

$$h_{ij} = \det\left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}, \frac{\partial^2 f}{\partial x^i \partial x^j}\right). \tag{1}$$

Because of convexity, the coordinates can be chosen so that h_{ij} is positive definite; then the metric is

$$g_{ij} = \frac{h_{ij}}{(\det h)^{1/(n+2)}}$$

(see [B]). Alternatively, given a transverse vector field N along the hypersurface, define a connection ∇, for vector fields X and Y along M, by splitting the ordinary derivative

$$D_X Y = g(X,Y)N + \nabla_X Y$$

into transverse and tangential parts. By wedge product with N, the invariant volume form on the ambient space gives a volume form Ω on M. Then N is the unique affine normal, and g is the affine metric, if Ω is parallel with respect to ∇ and coincides with the volume form of g (see [No]).

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Note that ∇ is not necessarily the Levi-Civita connection $\hat{\nabla}$ belonging to g. In fact, the next affine invariant for hypersurfaces is the Pick tensor, defined as the difference $\nabla - \hat{\nabla}$. When this $(2,1)$ tensor is lowered using g, the result is the Pick form, a cubic form on TM, with components A_{ijk} satisfying

$$A_{ijk} = -2\nabla_k g_{ij}, \qquad A_{ijk} = A_{jki} = A_{kij}, \qquad g^{ij} A_{ijk} = 0.$$

The famous Pick-Berwald theorem [B] says that M is a quadric hypersurface if and only if the Pick form vanishes identically.

The third affine invariant we will need is the shape operator associated to the affine normal N. One can show that the shape operator $S(X) = -\nabla_X N$ is a endomorphism of TM and that

$$B_{ij} = g_{ik} S^k{}_j$$

is symmetric; B is sometimes known as the affine second fundamental form.

The tensors g, A, B satisfy analogues of the Gauss and Codazzi equations (cf. [S]).

Proposition 1 (Gauss equations). *If g is the affine-invariant metric on a convex surface in $\mathbb{R}^3$, then the Gauss curvature of g satisfies*

$$K = (|A|^2 + g^{ij} B_{ij})/2. \tag{2}$$

Proposition 2 (Codazzi equations). *Under covariant differentiation using the Levi-Civita connection of g,*

$$B_{ij,k} - B_{ik,j} = A^m_{ij} B_{km} - A^m_{ik} B_{jm} \tag{3}$$

$$A_{ijk,l} - A_{ijl,k} = \tfrac{1}{2}(g_{ik} B_{jl} - g_{il} B_{jk} + g_{jk} B_{il} - g_{jl} B_{ik}). \tag{4}$$

Note that the Codazzi equations have the same form in higher dimensions.

In this article we will discuss the affine isometric embedding problem for surfaces:[1] that is, given a Riemannian metric g on a surface M, is there an immersion $f : M \to \mathbb{R}^3$, with strictly convex image, such that the affine invariant metric on the image coincides with g? It is well-known that the constant curvature surfaces can be isometrically embedded as quadric surfaces. (For example, any ellipsoid enclosing a volume $4\pi/3$ gets an affine invariant metric of constant curvature 1.) However, it is not known if there is any other way to embed these surfaces.

From a naive point of view, we are asking for the three components of f to satisfy a system of three second-order PDEs (1), and in this sense the problem is determined. At another level, we are asking for symmetric tensors A, B satisfying the Gauss and Codazzi equations. For, the fundamental theorem in affine surface theory then gives a local embedding into R^3:

[1] In [M], a claim to having solved the affine isometric embedding problem for surfaces is incorrectly attributed to the present author.

Radon's Theorem. ([B], [S]) *Let (M, g) be a smooth Riemannian surface, and let A, B be symmetric cubic and quadratic forms on M, such that A is traceless with respect to g, and A and B satisfy the above Gauss and Codazzi equations. Then given any point $p \in M$, there exists an isometric embedding f defined on a neighbourhood of p, which is unique up to affine motions.*

If M is simply connected, then f extends to all of M. Furthermore, if M is compact and oriented, then M is S^2 and the fact that f has a convex image implies that f is an embedding.

For the rest of this article, we will concentrate on embeddings of S^2 into R^3. Again, it is not known if there is any way of embedding the round sphere in any way except as an ellipsoid. It is also not known what other metrics on S^2 admit an affine isometric embedding. In this connexion, we obtain a noninvertibility result, which says that there are metrics arbitrarily close to the standard metric on S^2 which do not possess an affine isometric embedding near the ellipsoid.

2. Ellipticity

We will show here that the compatibility conditions for A and B can be reformulated as a second-order, elliptic system for the components of A. It is necessary to work at the level of the Pick form, because the original isometric embedding condition (1) fails to be elliptic. To see this, it is enough to work on the equation

$$h_{ij} = \left\{ \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} \right\},$$

since g can be recovered from h and vice-versa. (Here, the braces denote the scalar triple product.) Linearization of the right-hand side at an embedding f gives the following linear operator:

$$L(\tilde{f})_{ij} = \left\{ \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j} \right\} + \left\{ \frac{\partial \tilde{f}}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial^2 f}{\partial x^i \partial x^j} \right\} + \left\{ \frac{\partial f}{\partial x^1}, \frac{\partial \tilde{f}}{\partial x^2}, \frac{\partial^2 f}{\partial x^i \partial x^j} \right\}.$$

The symbol mapping associated to the top-order part of L is

$$\sigma_\xi(\tilde{f})_{ij} = \xi_i \xi_j \left\{ \frac{\partial f^\alpha}{\partial x^1}, \frac{\partial f^\beta}{\partial x^2}, \tilde{f} \right\},$$

which clearly has rank one for any covector ξ. In fact, since the first-order part of L, when restricted to the kernel of σ_ξ, also has rank one, L is not even elliptic in the generalized sense of Douglis-Nirenberg [D-N].

Suppose (M, g) is an oriented Riemannian surface. We will use the complex structure on M to express the Codazzi equations in a more compact form. Let η^1, η^2 be a local orthonormal coframing such that $\omega = \eta^1 + i\eta^2$

is a $(1,0)$-form. If A_{jkl} are the components of a traceless cubic form with respect to this coframing, then

$$A_{jkl}\eta^j\eta^k\eta^l = \mathbb{R}\mathrm{e}(A_{111} + iA_{222})\omega^3. \tag{5}$$

Let $\mathcal{T}$ denote the line bundle which is the $(1,0)$ part of $TM \otimes \mathbb{C}$, let $\mathcal{T}^{-1}$ denote the $(0,1)$ part, and let $\mathcal{T}^n$ denote the tensor powers of these bundles. Then ω is a local section of $\mathcal{T}$, and (5) says that we can regard the Pick form as the real part of a section of $\mathcal{T}^3$.

The bundle $\mathcal{T}$ and its tensor powers are all holomorphic line bundles with hermitian metrics defined by g. We will reformulate the Codazzi equations in terms of the canonical connections on these bundles. As in [Br], we split each covariant differentiation operator into its $(1,0)$ and $(0,1)$ pieces:

$$\partial : C^\infty(\mathcal{T}^m) \to C^\infty(\mathcal{T}^m \otimes \mathcal{T}), \qquad \bar\partial : C^\infty(\mathcal{T}^m) \to C^\infty(\mathcal{T}^m \otimes \mathcal{T}^{-1}).$$

We can use the canonical pairing $\omega^m \cdot \bar\omega^n = \omega^{m-n}$ to get

$$\partial : C^\infty(\mathcal{T}^m) \to C^\infty(\mathcal{T}^{m+1}), \qquad \bar\partial : C^\infty(\mathcal{T}^m) \to C^\infty(\mathcal{T}^{m-1}).$$

Note that while $\partial + \bar\partial$ is the usual splitting of the exterior derivative on functions, $\partial \circ \partial \neq 0$ in general. If $\sigma \in C^\infty(\mathcal{T}^m)$ is expressed locally as $s\omega^m$, then we calculate ∂ and $\bar\partial$ as follows: if ρ is a real-valued connection form satisfying $d\omega = i\rho \wedge \omega$, let

$$ds + ims\rho = s'\omega + s''\bar\omega; \tag{6}$$

then $s'\omega^{m+1}$ and $s''\omega^{m-1}$, where we interpret negative powers of ω as $\bar\omega$'s, give $\partial\sigma$ and $\bar\partial\sigma$ respectively.

Let $\alpha = (A_{111} + iA_{222})\omega^3$ be the complexified Pick form. Then the Gauss equation implies that $tr(S) = 2K - 4|\alpha|^2$. Since the traceless part of a quadratic form is the real part of a $(2,0)$ form on M, we will let

$$B_{jk}\eta^j\eta^k = (K - 2|\alpha|^2)g - \mathbb{R}\mathrm{e}(\beta)$$

for a section β of $\mathcal{T}^2$. Then the Codazzi equation (4) implies that

$$\bar\partial\alpha = \tfrac{1}{2}\beta.$$

In other words, g and α determine the affine second fundamental form.

Furthermore, the Codazzi equation (3) implies that

$$\bar\partial\beta = 2\alpha\bar\beta + 2\bar\alpha\partial\alpha - \partial K.$$

(The right-hand side gives a section of $\mathcal{T}$ once we use the canonical pairing.) Substituting for β in the last equation gives the following second order system of PDE for α:

$$\bar\partial^2\alpha = \bar\alpha\partial\alpha + 2\alpha\partial\bar\alpha - \tfrac{1}{2}\partial K. \tag{7}$$

If we choose conformal coordinates so that $\omega = dz$ at a point, then $\bar\partial = \partial/\partial\bar z$ plus lower-order terms. It follows that (7) is an elliptic system of PDE.

Now we have a new version of Radon's Theorem:

Theorem. *Let (M, g) be an oriented Riemannian surface, and let α be a smooth section of $\mathcal{T}^3$ satisfying (7). Then α is the Pick form for a local affine isometric embedding of M into $\mathbb{R}^3$, which is unique up to affine motions.*

Remarks 1. If K is constant, (7) becomes

$$\bar{\partial}^2\alpha = \bar{\alpha}\partial\alpha + 2\alpha\partial\bar{\alpha}, \tag{8}$$

with the solution $\alpha \equiv 0$ giving the embedding as a quadric surface. The question of whether the constant curvature metric on S^2 admits any other affine isometric embedding amounts to asking whether (8) admits any nontrivial solution.

2. When the isometric embedding problem is phrased in terms of finding tensors A, B satisfying (2),(3),(4), one may ask if these equations are even locally solvable: if not, perhaps there is a compatibility condition to be satisfied by g and its local Riemannian invariants. However, our rewriting these equations as a determined elliptic system of PDE shows that local solutions exist for any metric. (In fact, by applying the Cartan- Kähler theorem, we find that local solutions depend on an arbitrary choice of four functions of one variable.) Naturally, compatibility conditions come into play in higher dimensions; in an earlier paper [I], the author investigated local solvability for $n = 3$, showing that, generically, solutions are completely determined by the values of A and ∇A at one point.

3. Even when the equations are locally solvable, in higher dimensions characteristic classes give obstructions to globally realizing a metric from an affine embedding [BBG]. (I am indebted to the referee for this remark.)

3. Linearization

We will attempt to solve (7) by linearizing about the solution $\alpha \equiv 0$ for the standard metric g_0 of constant curvature $+1$ on the sphere. From now on, M will be S^2. Since every other metric on M is conformally equivalent to g_0, we will let

$$g = e^{2f} g_0$$

be an arbitrary metric on M, and we will rewrite (7) in terms of f and the covariant derivative operators associated to the background metric g_0.

Proposition. *Let $\partial_0, \bar{\partial}_0$ be the operators associated to g_0, and $\partial, \bar{\partial}$ those associated to g. Then (7) is equivalent to*

$$\bar{\partial}_0^2\alpha - 2\alpha\partial_0\bar{\alpha} - \bar{\alpha}(\partial_0\alpha - 6\alpha\partial f) + \tfrac{1}{2}\partial_0 K = 0, \tag{9}$$

where K is the Gauss curvature of g, given by

$$K = e^{-2f}(1 - 2\Delta f), \qquad \Delta = \partial\bar{\partial} + \bar{\partial}\partial.$$

(Note that ∂, $\bar\partial$, and Δ and the same as the corresponding operators for g_0 on functions.)

Sketch. Let ω_0 be a local $(1,0)$ form such that $g_0 = \omega_0\bar\omega_0$. Then $\omega = e^f\omega_0$ is the corresponding form for g, and the connection forms are related by

$$\rho = \rho_0 + i(\partial f - \bar\partial f).$$

Using (6), we now calculate that for a section σ of $\mathcal{T}^m$, provided $m > 0$,

$$\partial\sigma = \partial_0\sigma - 2m\sigma\partial f$$

$$\bar\partial\sigma = \bar\partial_0\sigma.$$

The recast equation (9) now follows easily.□

We will regard the left-hand side of (9) as a nonlinear differential operator on sections of $\mathcal{T}^3$. We can separate it into its linear and nonlinear parts:

$$P(\alpha) - R(\alpha, f) = 0, \tag{10}$$

where $P = \bar\partial_0^2$. We will extend these operators to the appropriate function spaces:

$$P : X \to Z, \qquad R : X \times Y \to Z$$

where $X = H_2^2(\mathcal{T}^3)$ (i.e. sections of $\mathcal{T}^3$ whose second derivatives are in L^2), Y is the Hölder space $C^{3,\delta}$ for some $\delta > 0$, and $Z = L^2(\mathcal{T})$.

In this setting, we might hope to employ an implicit function theorem to solve (10) for α as a function of f. This is not immediately feasable because P is not invertible:

Proposition. *The operator P has kernel zero, but its cokernel inside Z is nonzero, of complex dimension eight.*

Proof. P is an elliptic operator, and its adjoint $P^* : Z \to X$ is given by ∂_0^2, which is also elliptic. For each of these, the kernel will be finite-dimensional and smooth. In order to calculate the kernels, we need to calculate the kernel of $\bar\partial_0$ on $C^\infty(\mathcal{T}^m)$ for m both positive and negative.

Let z be the usual coordinate on S^2 *qua* the Riemann sphere. Then $\omega_0 = 2dz/(1 + z\bar z)$. Let $\sigma = s(\omega_0)^m$. Then $\bar\partial_0\sigma = 0$ if

$$ds + ims\rho_0 \equiv 0 \text{ mod } dz. \tag{11}$$

Using

$$\rho_0 = i\frac{zd\bar z - \bar z dz}{1 + z\bar z},$$

we see that (11) is equivalent to $\tilde s = (1+z\bar z)^{-m}s$ being a holomorphic function on $\mathbb{C}$.

If σ extends to all of S^2, then $|s|$ must be bounded. If $m > 0$, then $\bar{\partial}_0\sigma = 0$ implies that $\tilde{s}$ is zero at $z = \infty$, and so $\sigma = 0$. It follows that the kernel of P is zero. Furthermore, since $\bar{\partial}_0$ is the adjoint of the elliptic operator ∂_0, then $\partial_0 : C^\infty(\mathcal{T}) \to C^\infty(\mathcal{T}^2)$ must be surjective.

If $m < 0$, then $\tilde{s}$ can have at most a pole of order $|2m|$ at $z = \infty$. So, $\bar{\partial}_0\sigma = 0$ implies that $\tilde{s} = p(z)$ for some polynomial p of degree at most $|2m|$ in z. By complex conjugation, it follows that the kernels of ∂_0 on $C^\infty(\mathcal{T})$ and $C^\infty(\mathcal{T}^2)$ have complex dimension three and five, respectively. Because of the surjectivity of ∂_0, the kernel of P^* is isomorphic to the direct sum of these two kernels. □

When the linear part of (10) is noninvertible, but has zero kernel, we may still obtain a candidate solution by projecting into range(P), inverting, and then obtaining necessary conditions on f. (Nirenberg refers to this method as "bifurcation" [Ni]). To this end, recall that Z splits as an orthogonal direct sum

$$Z = \operatorname{range}(P) \oplus \ker(P^*),$$

and let $E : Z \to \operatorname{range}(P)$ be the projection. There exists a continuous left inverse $Q : \operatorname{range}(P) \to X$ for P, and applying E followed by Q to (10) gives

$$\alpha + Q \circ E \circ R(\alpha, f) = 0 \tag{12}$$

Theorem. *There exist neighbourhoods V and U of the origin in X and Y respectively, and a continuous map $J : U \to V$ such that $\alpha = J(f)$ is the unique solution of (12) in V for all $f \in U$.*

Proof. In order to apply the implicit function theorem (see [Ni], Ch. VII), we need only verify that R is continuous in α and f, and that R is actually of higher order in α. To check this, calculate

$$R(\alpha_1, f) - R(\alpha_2, f) = 2(\alpha_1\partial_0\bar{\alpha}_1 - \alpha_2\partial_0\bar{\alpha}_2) + \bar{\alpha}_1\partial_0\alpha_1 - \bar{\alpha}_2\partial_0\alpha_2 - 6(|\alpha_1|^2 - |\alpha_2|^2)\partial f.$$

Now suppose $|\alpha_1|, |\alpha_2|, |f| < \epsilon$. By interpolating terms, it is easy to see that

$$|R(\alpha_1, f) - R(\alpha_2, f)| = O(\epsilon)|\alpha_1 - \alpha_2|.$$

The existence of the map J now follows by the implicit function theorem. □

Now let $E' : Z \to \ker(L^*)$ be the other projection. In order for $\alpha = J(f)$ to be a solution of (10), we must have

$$E' \circ R(J(f), f) = 0.$$

However, linearizing this equation at $f = 0$ shows that it is not identically satisfied on U.

Proposition. *Let $S(f) = R(J(f), f)$ for $f \in U$, and let dS be the linearization of S at $f = 0$. Then $E' \circ dS \neq 0$.*

Proof. The linearization of R at $(0,0) \in X \times Y$ involves only the Gauss curvature term in (9), and is given by

$$dR(\tilde{f}) = \partial_0(\tilde{f} + \Delta\tilde{f}).$$

Then $E' \circ dS(\tilde{f}) = E'\partial_0(\tilde{f} + \Delta\tilde{f})$. To show that this is not zero, we must show that $\partial_0(\tilde{f} + \Delta\tilde{f})$ has a nonzero projection into $\ker P^*$ for some $\tilde{f}$. Suppose $\tilde{f}$ is a λ-eigenfunction for Δ. Taking the L^2 inner product with $\theta \in C^\infty(\mathcal{T})$ gives

$$\int_M < \theta, \partial_0(\tilde{f} + \Delta\tilde{f}) > dvol_{g_0} = \frac{\lambda+1}{2i}\int_M \bar{\theta} \wedge \partial_0\tilde{f}.$$

The eigenspaces of Δ are irreducible $SO(3)$-modules of dimensions $1, 3, 5, 7\ldots$, obtained by restricting harmonic polynomials on $\mathbb{R}^3$ of degrees $0, 1, 2, 3\ldots$ to S^2. On the other hand, $\ker P^*$ is the direct sum of two complex vector spaces which are, in fact, irreducible complex $SO(3)$-modules of dimensions three and five. When θ is in one of these modules, the above inner product gives an equivariant map from an eigenspace into the dual of the module. By Schur's Lemma, this map is either zero or an isomorphism.

Suppose, for example, $\tilde{f}$ is the restriction to S^2 of the linear function $ax_1 + bx_2 + cx_3$ on $\mathbb{R}^3$. Using stereographic projection,

$$\tilde{f} = \frac{2\mathbb{R}\mathrm{e}((a - ib)z - c)}{1 + z\bar{z}} + c,$$

and it is easy to check that $\tilde{f}$ is an eigenfunction for $\lambda = -4$. Then

$$\partial_0\tilde{f} = \frac{a - ib + 2c\bar{z} - (a + ib)\bar{z}^2}{1 + z\bar{z}}\omega_0.$$

In the proof of the previous proposition, we've explained how to write down a $(1,0)$-form θ in the kernel of ∂_0, in terms of an arbitrary second degree polynomial in $\bar{z}$. With the obvious choice, the inner product with the above $\partial_0\tilde{f}$ is nonzero. Similarly, one can verify that the inner product using functions in the five-dimensional eigenspace ($\lambda = -12$) with the appropriate forms θ is nonzero. □

Corollary. *Given any neighbourhood $\tilde{U}$ of the origin in $C^\infty(S^2)$, there exist functions $f \in \tilde{U} \cap U$ with no solution to (10) in V. In other words, the corresponding metrics $g = e^{2f}g_0$ do not have affine isometric embeddings near the ellipsoid.*

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