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Hypersurfaces with Codazzi-type shape operator for a Tanaka–Webster connection

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Abstract. Perez and Suh have classified the hypersurfaces in complex projective space $\mathbb{C}\mathrm{P}^n$ which satisfy the condition given in the title, under the assumption that $n \geq 3$. In this paper, we complete the classification by proving the same result for the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ as well as for $\mathbb{C}\mathrm{P}^n$. Our proof holds for all $n \geq 2$.

1. Introduction

The Tanaka–Webster connection [12], [15] originally occurred in the study of pseudo-Hermitian CR-manifolds, and was later extended by Tanno [13] to contact metric manifolds. It has recently been generalized to hypersurfaces in complex space forms by Cho, who introduced the generalized Tanaka–Webster (abbreviated g-Tanaka–Webster) connection and studied hypersurfaces that are parallel [3], or that have constant holomorphic sectional curvature [4] with respect to this connection.

The Hopf hypersurfaces in $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{H}^n$ that have constant principal curvatures are open subsets of certain model spaces (Takagi's list [11] and Montiel's list [7], described in more detail below) which have a well-established nomenclature:

- Types A_1, A_2, B, C, D and E in $\mathbb{C}\mathrm{P}^n$ (Takagi's list)
- Types A_0, A_1, A_2, and B in $\mathbb{C}\mathrm{H}^n$ (Montiel's list).

Types A_0, A_1 and A_2 are referred to collectively as Type A.

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Perez and Suh [9] proved

Theorem 1. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in $\mathbb{C}\mathrm{P}^n$ whose shape operator is of Codazzi type with respect to a g-Tanaka–Webster connection. Then M must be a Hopf hypersurface.*

and

Theorem 2. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in $\mathbb{C}\mathrm{P}^n$ and let $\hat{\nabla}^{(k)}$ be a g-Tanaka–Webster connection for which $2k \neq \alpha$. Then M is of Codazzi type with respect to $\hat{\nabla}^{(k)}$ if and only if M is an open subset of a Type A hypersurface.*

Here α is the Hopf principal curvature and k is the real number determining the particular g-Tanaka–Webster connection. This connection will be defined in §3. For a detailed explanation of its relationship to the theory of CR-manifolds, the induced almost contact metric structure and the constructions of Tanaka, Webster, and Tanno, the interested reader is referred to the review of [9] by S. Dragomir in *Mathematical Reviews* (MR3150837).

The purpose of this paper is to complete the classification of Perez and Suh by proving that the same results hold for both ambient spaces, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{H}^n$, and for all $n \geq 2$ (see Theorems 5, 7 and 8). The authors are grateful to the referee who pointed out a deficiency in an earlier version of our proof of Theorem 7.

In our exposition, all manifolds are assumed connected and all manifolds and maps are assumed smooth (C^∞) unless stated otherwise.

2. Basic equations and results for hypersurfaces

We follow the notation and terminology of [8]. M^{2n-1} will be a hypersurface in a complex space form, either $\mathbb{C}\mathrm{P}^n$ or $\mathbb{C}\mathrm{H}^n$, of constant holomorphic sectional curvature $4c = \pm 4/r^2$. The locally defined field of unit normals is ξ, the structure vector field is $W = -J\xi$ and φ is the tangential projection of the complex structure J. The holomorphic distribution consisting of all tangent vectors orthogonal to W is denoted by $W^\perp$ and $\varphi^2 X = -X$ for all $X \in W^\perp$.

The shape operator A of M is defined by

$$AX = -\widetilde{\nabla}_X \xi$$

where $\widetilde{\nabla}$ is the Levi–Civita connection of the ambient space and X is any tangent vector to M. (It follows that $\nabla_X W = \varphi A X$, see [8] p. 239.) The eigenvalues of A

are the principal curvatures and the corresponding eigenvectors and eigenspaces are said to be principal vectors and principal spaces. The function $\langle AW, W\rangle$ is denoted by α. If W is a principal vector at all points of M (and so $AW = \alpha W$), we say that M is a *Hopf hypersurface* and α is called the Hopf principal curvature. For a Hopf hypersurface, the Hopf principal curvature is constant. We state the following fundamental facts (see Corollary 2.3 of [8]).

Lemma 3. *Let M be a Hopf hypersurface and let $X \in W^{\perp}$ be a principal vector with associated principal curvature λ. Then*

(1) $\left(\lambda - \frac{\alpha}{2}\right)A\varphi X = \left(\frac{\lambda\alpha}{2} + c\right)\varphi X$;

(2) *If $A\varphi X = \nu\varphi X$ for some scalar ν, then $\lambda\nu = \frac{\lambda+\nu}{2}\alpha + c$;*

(3) *If $\nu = \lambda$ in (2), then $\lambda^2 = \alpha\lambda + c$.*

Takagi's list and Montiel's list. The Takagi/Montiel list consists precisely of the complete Hopf hypersurfaces with constant principal curvatures in their respective ambient spaces as determined by Kimura [6] and Berndt [1]. Equivalently, it is the list of homogeneous Hopf hypersurfaces, a fact that follows from the work of Takagi [10] and Berndt [1]. Non-Hopf homogeneous hypersurfaces exist in $\mathbb{C}\mathrm{H}^n$ but not in $\mathbb{C}\mathrm{P}^n$.

Takagi's list for $\mathbb{C}\mathrm{P}^n$

- (A_1) Geodesic spheres (which are also tubes over totally geodesic complex projective spaces $\mathbb{C}\mathrm{P}^{n-1}$).
- (A_2) Tubes over totally geodesic complex projective spaces $\mathbb{C}\mathrm{P}^k, 1 \leq k \leq n-2$.
- (B) Tubes over complex quadrics (which are also tubes over totally geodesic real projective spaces $\mathbb{R}P^n$).
- (C) Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$ where $2m+1 = n$ and $n \geq 5$.
- (D) Tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}$ (which occur only for $n = 9$).
- (E) Tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ (which occur only for $n = 15$).

Note that when $n = 2$, the only types that occur are A_1 and B.

Montiel's list for $\mathbb{C}\mathrm{H}^n$

- (A_0) Horospheres.
- (A_1) Geodesic spheres and tubes over totally geodesic complex hyperbolic spaces $\mathbb{C}\mathrm{H}^{n-1}$.

- (A_2) Tubes over totally geodesic complex hyperbolic spaces $\mathbb{C}\mathrm{H}^k, 1 \le k \le n-2$.
- (B) Tubes over totally geodesic real hyperbolic spaces $\mathbb{R}\mathrm{H}^n$.

Note that Type A_2 cannot occur when $n = 2$.

3. The *g*-Tanaka–Webster connections

Let M^{2n-1} be a real hypersurface in $\mathbb{C}\mathrm{P}^n$ or $\mathbb{C}\mathrm{H}^n$ where $n \geq 2$. For a nonzero real constant k, we define the g-Tanaka–Webster connection by the formula

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + \langle \varphi AX, Y\rangle W - \langle W, Y\rangle \varphi AX - k\langle W, X\rangle \varphi Y.$$

For motivation and further explanation of this definition, see [3] and [5].

To say that A is a Codazzi tensor (or of Codazzi type) with respect to $\hat{\nabla}^{(k)}$ means that for all tangent vectors X and Y,

$$\left(\hat{\nabla}^{(k)}_X A\right) Y - \left(\hat{\nabla}^{(k)}_Y A\right) X = 0. \tag{1}$$

In view of the Codazzi equation for hypersurfaces in complex space forms (see [8], equation (1.9)), this is equivalent to the following condition:

$$\begin{aligned} & c(\langle X, W\rangle \varphi Y - \langle Y, W\rangle \varphi X + 2\langle X, \varphi Y\rangle W) \\ &\quad = -2\langle A\varphi AX, Y\rangle W + \langle (\varphi A + A\varphi)X, Y\rangle AW \\ &\qquad - \langle W, AX\rangle \varphi AY + \langle W, AY\rangle \varphi AX + \langle W, X\rangle A\varphi AY - \langle W, Y\rangle A\varphi AX \\ &\qquad + k(\langle W, X\rangle(\varphi A - A\varphi)Y - \langle W, Y\rangle(\varphi A - A\varphi)X). \end{aligned} \tag{2}$$

This is the same as equation (3.2) in [9] if we set $c = 1$ and make the obvious translation of notation.

Taking the inner product of (2) with W yields

$$\begin{aligned} & 2c\langle X, \varphi Y\rangle + 2\langle A\varphi AX, Y\rangle - \alpha\langle (\varphi A + A\varphi)X, Y\rangle \\ &= -\langle W, X\rangle\langle Y, AV\rangle + \langle W, Y\rangle\langle X, AV\rangle - k\langle W, Y\rangle\langle X, V\rangle + k\langle W, X\rangle\langle Y, V\rangle, \end{aligned} \tag{3}$$

where $V = \varphi AW$. Because this holds for all Y, we have

$$2\left(A\varphi A - \frac{\alpha}{2}(\varphi A + A\varphi) - c\varphi\right) X = \langle X, (A-k)V\rangle W - \langle W, X\rangle (A-k)V. \tag{4}$$

Now take the inner product of (4) with W. Since $\langle (A-k)V, W\rangle = \langle V, (A-k)W\rangle = \langle \varphi AW, (A-k)W\rangle = 0$, we obtain

$$2\left\langle X, \left(A - \frac{\alpha}{2}\right)V\right\rangle + \langle X, (A-k)V\rangle = 0$$

for all X, i.e.

$$AV = \frac{1}{3}(\alpha + k)V. \tag{5}$$

Thus (4) becomes

$$2\left(A\varphi A - \frac{\alpha}{2}(\varphi A + A\varphi) - c\varphi\right)X = \frac{1}{3}(\alpha - 2k)(\langle X, V\rangle W - \langle W, X\rangle V). \tag{6}$$

In particular, taking $X = V$ and using (5) yields

$$(\alpha - 2k)(A\varphi V + |V|^2 W) + (\alpha^2 + k\alpha + 6c)\varphi V = 0. \tag{7}$$

On the other hand, if we set $X = W$ and $Y = V$ in (2), we get

$$(\alpha - 2k)(A\varphi V + |V|^2 W) = (3c + \alpha^2 - k^2 + 3|V|^2)\varphi V \tag{8}$$

where we have used (5) and the fact that $AW - \alpha W = -\varphi V$, so that $\langle A\varphi V, W\rangle = -|V|^2$. Thus, if $V \neq 0$, we have

$$2\alpha^2 + k\alpha - k^2 + 9c + 3|V|^2 = 0. \tag{9}$$

Lemma 4. *The span of $\{W, AW\}$ is A-invariant.*

Proof. This is clearly true if $AW = \alpha W$ so we assume that $\beta = |AW - \alpha W| = |V| \neq 0$ and let U be the unit vector such that $\varphi V = -\beta U$. Then $AW = \alpha W + \beta U$. Also, $\{W, AW\}$ and $\{W, U\}$ have the same span.

If $\alpha - 2k \neq 0$, then AU is a linear combination of U and W by (7). On the other hand, if $\alpha = 2k$, then (7) gives $\alpha^2 + k\alpha + 6c = 0$, which reduces to $\alpha^2 + 4c = 0$. However, (8) now gives $0 = 3c + \alpha^2 - k^2 + 3\beta^2 = 3c - 4c + c + 3\beta^2$, which contradicts $V \neq 0$. □

4. The case $n \geq 3$

We continue to assume that W is not principal. By Lemma 4 and the fact that V is principal, the orthogonal complement to the span of $\{W, U, V\}$ is also A-invariant. Suppose that X is a unit principal vector orthogonal to this span and that X corresponds to a principal curvature $\lambda \neq \frac{\alpha}{2}$. Then (6) shows that φX is also principal and the corresponding principal curvature ν satisfies the formula familiar from the study of Hopf hypersurfaces (see Lemma 3)

$$\lambda\nu = \frac{\lambda+\nu}{2}\alpha + c. \tag{10}$$

However, if we evaluate (2) with this particular X and with $Y = W$, we get

$$-c\varphi X = \alpha\lambda\varphi X - \lambda\nu\varphi X - k(\lambda-\nu)\varphi X$$

which, using (10), simplifies to

$$(\lambda-\nu)(\alpha-2k) = 0.$$

Since we have already shown that $\alpha = 2k$ leads to a contradiction, we conclude that $\lambda = \nu$.

Now consider (2) with $Y = -\varphi X$ (so that $X = \varphi Y$). Most terms are zero, but the remaining ones give

$$2cW = 2\lambda^2 W - 2\lambda AW. \tag{11}$$

This contradicts our assumption that W is not a principal vector. We conclude that if X is any unit principal vector orthogonal to the span of $\{W, U, V\}$, the associated principal curvature must be $\frac{\alpha}{2}$. However, if we now evaluate (2) with $Y = -\varphi X$ and $\lambda = \nu = \frac{\alpha}{2}$, we get the same equation (11) which is again a contradiction.

We conclude that there are no principal vectors orthogonal to the span of $\{W, U, V\}$, i.e. a non-Hopf hypersurface satisfying (1) cannot occur for $n \geq 3$. Thus, we have proved the following theorem which extends the first theorem of Perez and Suh to cover $\mathbb{C}\mathrm{H}^n$ as well as $\mathbb{C}\mathrm{P}^n$.

Theorem 5. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in $\mathbb{C}\mathrm{P}^n$ or $\mathbb{C}\mathrm{H}^n$ whose shape operator is of Codazzi type with respect to a g-Tanaka–Webster connection. Then M must be a Hopf hypersurface.*

5. The case $n = 2$

If we review our previous analysis, restricting ourselves to the case $n = 2$, everything in §3 applies. We assume again that $AW \neq \alpha W$, and consequently $\alpha -$

$2k$ is nonvanishing. The triple of vectors $(W, U, \varphi U)$ constitutes an orthonormal basis for the tangent space and the shape operator can be expressed relative to this basis by the matrix

$$\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \tag{12}$$

where $\beta \neq 0$ and $\nu = (\alpha + k)/3$.

Taking $X = U$ and $Y = \varphi U$ in (2), we get

$$(\lambda + \nu)AW = -\nu\beta U + 2(\lambda\nu - c)W.$$

Since $\langle AW, U\rangle = \beta$, we get $\lambda = -2\nu$. Similarly, taking inner product with W, we get $(\lambda + \nu)\alpha = 2(\lambda\nu - c)$, which now reduces to

$$4\nu^2 - \nu\alpha + 2c = 0. \tag{13}$$

Expressing ν in terms of α and k, we get

$$\alpha^2 + 5\alpha k + 4k^2 + 18c = 0. \tag{14}$$

Thus ν and λ are nonzero constants and α is also constant. But we also have the following result, which will be proved below:

Lemma 6. *Any hypersurface in $\mathbb{C}\mathrm{P}^2$ or $\mathbb{C}\mathrm{H}^2$ whose shape operator has the form* (12) *with α and ν constant and $\lambda = -2\nu$, also has β constant.*

Thus M has constant principal curvatures. In $\mathbb{C}\mathrm{P}^2$ this is already a contradiction, since all hypersurfaces with constant principal curvatures must be Hopf, as was shown long ago by Q. M. Wang [14]. For $\mathbb{C}\mathrm{H}^2$, according to the classification of Berndt and Díaz-Ramos [2], M must be an open subset of a standard homogeneous hypersurface. However, for such a hypersurface, the trace of A is 4ν while that of M is $\alpha - \nu$. For these to be the same, we would need $\alpha = 5\nu$ which is clearly inconsistent with $4\nu^2 - \nu\alpha + 2c = 0$. Thus, the assumption that M is not Hopf has led to a contradiction, and we have proved

Theorem 7. *Let M^3 be a real hypersurface in $\mathbb{C}\mathrm{P}^2$ or $\mathbb{C}\mathrm{H}^2$ whose shape operator is of Codazzi type with respect to a g-Tanaka–Webster connection. Then M must be a Hopf hypersurface.*

Proof of Lemma 6. Let $\widetilde{V} = \varphi U$ so that $V = \varphi AW = \beta\widetilde{V}$. We will use the Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X = c(\langle X, W\rangle\varphi Y - \langle Y, W\rangle\varphi X + 2\langle X, \varphi Y\rangle W)$$

for a hypersurface in a complex space form (see [8] p. 238), along with special properties of our basis vectors W, U, $\widetilde{V}$, to derive formulas for covariant derivatives of the basis vectors in terms of the coefficients in the matrix (12). We group the computations into steps [i]–[iv]:

[i] $\nabla_W W = \beta\widetilde{V}$, $\nabla_U W = \lambda\widetilde{V}$ and $\nabla_{\widetilde{V}} W = -\nu U$.

These follow immediately from the identity $\nabla_X W = \varphi AX$.

[ii] $\nabla_W \widetilde{V} = (\alpha - 3\nu)U - \beta W$ and $\nabla_{\widetilde{V}}\widetilde{V} = 0$.

Consider the Codazzi equation with $X = \widetilde{V}$ and $Y = W$. Its right side is cU, while its left side is the difference of

$$\begin{aligned}(\nabla_{\widetilde{V}} A)W &= \nabla_{\widetilde{V}}(\alpha W + \beta U) - A\nabla_{\widetilde{V}} W = -\nu(\alpha - A)U + (\widetilde{V}\beta)U + \beta\nabla_{\widetilde{V}} U \\ &= -\nu(\alpha - \lambda)U + \nu\beta W + (\widetilde{V}\beta)U + \beta\nabla_{\widetilde{V}} U\end{aligned}$$

and

$$(\nabla_W A)\widetilde{V} = (\nu - A)\nabla_W \widetilde{V}.$$

Taking the inner product of the Codazzi equation with W, we get

$$\begin{aligned}\nu\beta + \beta\langle\nabla_{\widetilde{V}} U, W\rangle &= \langle(\nu - A)\nabla_W \widetilde{V}, W\rangle \\ \nu\beta - \beta\langle U, -\nu U\rangle &= \langle\nabla_W \widetilde{V}, (\nu - \alpha)W\rangle + \langle\nabla_W \widetilde{V}, -\beta U\rangle \\ 2\nu\beta &= -(\nu - \alpha)\langle\widetilde{V}, \beta\widetilde{V}\rangle - \beta\langle\nabla_W \widetilde{V}, U\rangle.\end{aligned}$$

Thus $\langle U, \nabla_W \widetilde{V}\rangle = \alpha - 3\nu$, so that

$$\nabla_W \widetilde{V} = (\alpha - 3\nu)U - \beta W.$$

On the other hand, the inner product of the Codazzi equation with $\widetilde{V}$ yields $\langle\nabla_{\widetilde{V}} U, \widetilde{V}\rangle = 0$. Since we already know that

$$\langle\nabla_{\widetilde{V}}\widetilde{V}, W\rangle = -\langle\widetilde{V}, \nabla_{\widetilde{V}} W\rangle = 0,$$

we have $\nabla_{\widetilde{V}}\widetilde{V} = 0$.

[iii] $\nabla_U U = \tau\widetilde{V}$ where τ satisfies $\beta\tau = (\beta^2 - c) + 5\nu\alpha - 11\nu^2$.

Consider the Codazzi equation with $X = U$ and $Y = W$. Then $(\nabla_U A)W$ reduces to

$$\lambda(\alpha - \nu)\widetilde{V} + \beta\nabla_U U + (U\beta)U$$

while $(\nabla_W A)U$ is

$$(\lambda - A)\nabla_W U + (W\beta)W + \beta^2\widetilde{V}.$$

The right side is $-c\widetilde{V}$. Taking the inner product with $\widetilde{V}$ yields $\beta\tau = (\beta^2 - c) + 5\nu\alpha - 11\nu^2$ where $\tau = \langle\nabla_U U, \widetilde{V}\rangle$ and we have used the fact that $\lambda = -2\nu$. Since

$$\langle\nabla_U U, W\rangle = -\langle U, \nabla_U W\rangle = 0,$$

we have $\nabla_U U = \tau\widetilde{V}$.

[iv] $\tau = 0$.

Consider the Codazzi equation with $X = U$ and $Y = \widetilde{V}$. We can use our knowledge of $\nabla_U U$ and $\nabla_U W$ to get $\nabla_U \widetilde{V} = -\lambda W - \tau U$. Then

$$(\nabla_U A)\widetilde{V} = -(\nu - A)(\lambda W + \tau U) = -\lambda(\nu - \alpha)W + \lambda\beta U - \tau(\nu - \lambda)U + \tau\beta W$$

$$= (\alpha\lambda - \lambda\nu + \tau\beta)W + (\lambda\beta - \tau\nu + \lambda\tau)U = (2\nu^2 - 2\nu\alpha + \tau\beta)W - (2\nu\beta + 3\tau\nu)U$$

and

$$\begin{aligned}(\nabla_{\widetilde{V}} A)U &= (\lambda - A)\nabla_{\widetilde{V}} U + (\widetilde{V}\beta)W + \beta\nabla_{\widetilde{V}} W \\ &= \nu(\lambda - \alpha)W - \nu\beta U + (\widetilde{V}\beta)W - \beta\nu U \\ &= -(2\nu^2 + \nu\alpha)W - \nu\beta U + (\widetilde{V}\beta)W - \beta\nu U,\end{aligned}$$

where we have used the fact that $\nabla_{\widetilde{V}} U = \nu W$. This is clear because $\nabla_{\widetilde{V}} W = -\nu U$ and $\nabla_{\widetilde{V}} \widetilde{V} = 0$. The right side of the Codazzi equation is $-2cW$. Now take the inner product of both of these expressions with U to get $-2\nu\beta - 3\tau\nu + 2\nu\beta = 0$, i.e. $\tau = 0$.

Thus, from [iii] and [iv] we see that β is constant. $\square$

Remark. For further details on hypersurfaces with constant principal curvatures, we refer the reader to Proposition 3.5 of [2]. In the notation of Berndt and Díaz-Ramos, there is one principal direction orthogonal to W with principal curvature λ_3 and the sum of the other two principal curvatures is $3\lambda_3$. Since their λ_3 is our ν, this justifies our statement that A has trace equal to 4ν.

Both [14] and [2] assume three distinct constant principal curvatures. However, this is implied by our situation, since if ν were to have multiplicity 2, we would have, by (12),

$$(\alpha - \nu)(\lambda - \nu) - \beta^2 = 0.$$

Comparing this with our determination of β^2 from [iii] and [iv] gives $8\nu^2 - 2\nu\alpha + c = 0$ which contradicts (13).

6. Hopf hypersurfaces with $\hat{\nabla}^{(k)}$-Codazzi shape operator

Suppose now that M is a Hopf hypersurface satisfying (1). Then, the analysis of §4 and §5 holds with $|V| = \beta = 0$. In particular, (6) gives

$$\left(A\varphi A - \frac{\alpha}{2}(\varphi A + A\varphi) - c\varphi\right) X = 0 \tag{15}$$

for all $X \in W^\perp$. If $X \in W^\perp$ is a unit principal vector with $AX = \lambda X$ where $\lambda \neq \frac{\alpha}{2}$, we have $A\varphi X = \nu\varphi X$ where λ and ν satisfy (10). On the other hand, if we set $Y = W$ in (2), we obtain

$$-c\varphi X = \alpha\lambda\varphi X - \lambda\nu\varphi X - k(\lambda - \nu)\varphi X$$

which simplifies to

$$\left(\frac{\alpha}{2} - k\right)(\lambda - \nu) = 0.$$

Now k is constant and, since M is a Hopf hypersurface, α is also constant. If $k \neq \frac{\alpha}{2}$, we have $\lambda = \nu$ so that $\lambda^2 = \alpha\lambda + c$ and hence there are at most two values for λ. Consider the possibility that there are three distinct principal curvatures λ_1, λ_2 and $\frac{\alpha}{2}$ corresponding to principal vectors in $W^\perp$. Then there is a neighborhood in M which is a Hopf hypersurface with these particular constant values for the principal curvatures. However, this is impossible since there is nothing in the lists of Takagi and Montiel with these data. (For principal curvature information on these hypersurfaces, see [8], pp. 257–261.)

A second possibility is that we have principal curvatures λ_1 and λ_2 as in the previous paragraph, but their principal spaces span $W^\perp$. Then, there is a neighborhood $\mathcal{U}$ with the same constant principal curvatures and multiplicities. Then $\mathcal{U}$ must be an open subset of a Type A_2 hypersurface. For any specific choice of λ_1 and λ_2 (and associated multiplicities), the set of points of M matching these data is open and closed and is hence all of M.

This exhausts the possibility of the existence of a point with two distinct values of λ. If there is a point where there is one value of λ, there cannot also be a principal curvature equal to $\frac{\alpha}{2}$, by an argument similar to the one used for the $\left(\lambda_1, \lambda_2, \frac{\alpha}{2}\right)$ possibility. Thus, there is a neighborhood $\mathcal{U}$ where there is one constant principal curvature whose principal space is $W^\perp$. Then $\mathcal{U}$ must be an open subset of a Type A_1 hypersurface, and by the same connectedness argument, M itself is an open subset the same hypersurface.

The only remaining possibility is that for every point of M, $\frac{\alpha}{2}$ is a principal curvature whose principal space is $W^\perp$. Then M must be an open subset of a horosphere.

Finally, for any constant $k \neq 0$, and every Type A hypersurface M in $\mathbb{C}\mathrm{P}^n$ or $\mathbb{C}\mathrm{H}^n$, the shape operator is of Codazzi type with respect to the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$. This is straightforward to check using (2) together with the observation that $W^\perp$ consists of one or two A-invariant eigenspaces.

Thus, we have the following theorem.

Theorem 8. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in $\mathbb{C}\mathrm{P}^n$ or $\mathbb{C}\mathrm{H}^n$ and let $\hat{\nabla}^{(k)}$ be a g-Tanaka–Webster connection for which $2k \neq \alpha$. Then M is of Codazzi type with respect to $\hat{\nabla}^{(k)}$ if and only if M is an open subset of a Type A hypersurface.*

Remark. One can check that for any Hopf hypersurface with $\alpha \neq 0$, the shape operator is of Codazzi type with respect to one particular g-Tanaka–Webster connection, namely the one determined by $k = \frac{\alpha}{2}$. Thus, the specification that $2k \neq \alpha$ is necessary to characterize the Type A hypersurfaces. This should be taken into account in interpreting the statement of Theorem 1.2 in [9].

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